Exercise 7

1. Find all minimum/maximum points of the function

$$F(x,y) = xy - x - y + 3 \ ,$$

over the triangle with vertices at $(0,0),(2,0)$, and $(0,4)$.

2. Find all maximum/minimum points of the function

$$z = xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \ ,$$

in its natural domain.

3. Determine whether the following problems have maximum or minimum in $\mathbb{R}^2$. Not need to find them.

   (a) $g(x,y) = x^3 + y^3 - 3xy$,
   (b) $h(x,y) = x^4 + y^4 - x^2 - xy - y^2$,
   (c) $k(x,y) = \sin xy + x^2$.
   (d) $f(x,y) = (x - y + 1)^2$,

   Suggestion: Theorem 7.2 and its corollaries would be useful.

4. Find all maximum/minimum points of the function $u = x^2 - xy + y^2 - 2x + y$.

5. Find all maximum/minmimum points of the function $u = xy^2(1 - x - 2y), \quad x, y > 0$.

6. Let $(x_1, y_1), \cdots, (x_n, y_n)$ be n many order pairs. Find the straight line $y = ax + b$ so that the square error

$$F(a,b) = \sum_{j=1}^{n} (y_j - ax_j - b)^2 \ , \quad (a,b) \in \mathbb{R}^2,$$

is minimized.

7. Find all maximum/minimum points of the function

$$w(x,y) = xy + \frac{50}{x} + \frac{20}{y} \ , \quad x, y > 0 \ .$$

8. Find and classify the critical points of the following functions

   (a) $f_1(x,y) = 9 + 4y - 3x^2 - 2y^2 + 4xy$.
   (b) $f_2(x,y) = 3x - x^3 - 3xy^2$.
   (c) $f_3(x,y) = x^4 + y^4 - 4xy$.

9. Find and classify the critical points of the function
$$H(x,y) = xy\log(x^2+y^2), \quad (x,y) \neq (0,0) \ ,$$
and $H(0,0) = 0$.

10. Find all maximum/minimum points of the function
$$h(x,y,z) = xyz \ ,$$
over the set $\{(x,y,z) : x+y+z = 1, \ x,y,z \geq 0 \ \}$.

11. Consider a hexagon with vertices $(\pm 1, 0), (\pm x, \pm y), x, y \geq 0$, inscribed in the unit circle. Show that the area is maximal when it is a regular hexagon with equal sides and angles.

12. Use Lagrange multiplier to show that the distance from a point z to the hyperplane $H : a \cdot x = b$ is given by
$$\frac{|a \cdot z - b|}{|a|} \ .$$

13. Find the points on the ellipse $x^2 + xy + y^2 = 3$ that are closest to and farthest from the origin. Hint: Write the equations in the form $ax + by = 0, \quad cx + dy = 0$, and use the fact that $ad - bc = 0$ if there are non-trivial solutions.

14. Let $p \in G$ be a critical point of f subject to $g = 0$. Show that if its Lagrange multiplier is non-zero, then it is also a critical point of g subject to $f - f(p) = 0$.

15. (a) Find the points of the hyperbola $xy = 1$ that are closest to the origin $(0,0)$.
    (b) Show that the same points maximize xy over the circle $x^2 + y^2 - 2 = 0$.
    (c) Can you explain the "duality" in (a) and (b)?

16. A company uses the Cobb-Douglas production function
$$N(x,y) = 10x^{0.6}y^{0.4}$$
to estimate a new product. Here x is the number of units of labor and y is the number of units of capital required to produce $N(x,y)$ units of the product. Each unit of labor costs \$30 and each unit of capital costs \$60. If \$300,000 is budgeted for the production, determine how that amount should be allocated to maximize production, and find the maximum production.

17. Let T be a right triangle with sides x, y and hypotenuse z. Find the one maximizing the area subject to the perimeter constraint $x + y + z = 10$. Does there exist an area minimizing one?

18. Find the maximum/minimum points of the function $g(x,y,z) = xy + z^2$ subject to the constraints $y - x = 0$ and $x^2 + y^2 + z^2 = 4$.